

A Characterization of Substar Graphs

Felix Joos*

Abstract

The intersection graphs of stars in some tree are known as substar graphs. In this paper we give a characterization of substar graphs by the list of minimal forbidden induced subgraphs. This corrects a flaw in the main result of Chang, Jacobson, Monma and West (Subtree and substar intersection numbers, Discrete Appl. Math. **44**, 205-220 (1993)) and this leads to a different list of minimal forbidden induced subgraphs.

Keywords: chordal graph, intersection graph, substar graph

1 Introduction

A graph G is *chordal*, if it has no induced cycle of order at least four. Chordal graphs are one of the most fundamental graph classes and well investigated [1]. For a family \mathcal{M} of sets, an \mathcal{M} -*intersection representation* of a graph G is a function $f : V(G) \rightarrow \mathcal{M}$ such that two distinct vertices x and y are adjacent if and only if $f(x) \cap f(y) \neq \emptyset$. If G has an \mathcal{M} -intersection representation, then G is an \mathcal{M} -*graph*.

It is a classic result [6] in graph theory that every chordal graph is a \mathcal{T} -graph, where \mathcal{T} is the set of all subtrees of some tree. There are several important and well investigated subclasses of chordal graphs. Let \mathcal{T}_P be the set of all paths in some tree. We call a \mathcal{T}_P -graph a *path graph*. Path graphs are investigated in many different varieties [5, 7]. If \mathcal{T}_I is the set of all subpaths of some path, then the class of \mathcal{T}_I -graphs are known as *interval graphs*, which are another very important graph class [4].

A *star* is a tree such that there is a vertex that is adjacent to all other vertices of the tree. Let \mathcal{T}_S be the set of all substars in some tree T . We call a \mathcal{T}_S -graph a *substar graph*. Chang et al. [3] claimed to characterize substar graphs by a finite list of forbidden induced subgraphs. Since there are infinitely many minimal forbidden induced subgraphs for substar graphs, their claim is false. In this paper we give a characterization of substar graphs by the infinite list of minimal forbidden induced subgraphs.

Cerioli and Szwarcfiter [2] characterized a subclass of substar graphs, which is the class of starlike graphs. These are intersection graphs of substars of a star.

We start by introducing some basic notation. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G , respectively. A subset Q of the vertices of

*Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany, e-mail: felix.joos@uni-ulm.de

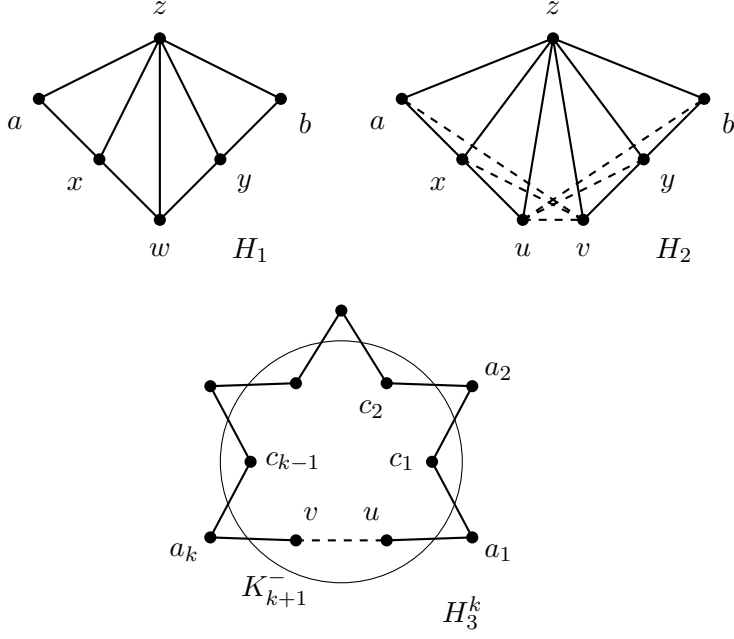


Figure 1: The set \mathcal{S} of minimal forbidden induced subgraphs for substar graphs.

G is a *clique* in G if every pair of vertices of Q are adjacent, and Q is a *maximal clique* in G if it is a clique not properly contained in another clique of G . Let the number of edges in a longest shortest path in G be the *diameter* of G . For $k \in \mathbb{N}$, we write $[k]$ for the set $\{1, \dots, k\}$.

2 Results

Before we state and prove our main result, we first introduce the set \mathcal{S} of minimal forbidden induced subgraphs for substar graphs. See Figure 1 for an illustration. The graph H_1 needs no further explanation. The graphs of type H_2 are as in Figure 1 and the dashed edges indicate the following three possibilities:

- $N_{H_2}(u) \cap \{v, y, b\} = N_{H_2}(v) \cap \{u, x, a\} = \emptyset$,
- $N_{H_2}(u) \cap \{v, y, b\} = \{v\}$ and $N_{H_2}(v) \cap \{u, x, a\} = \{u, x, a\}$, or
- $N_{H_2}(u) \cap \{v, y, b\} = \{v, y, b\}$ and $N_{H_2}(v) \cap \{u, x, a\} = \{u, x, a\}$.

The graph H_3^k for some $k \geq 3$ satisfies $V(H_3^k) = \{u, v, c_1, \dots, c_{k-1}, a_1, \dots, a_k\}$ and

$$E(H_3^k) = \left(\binom{\{u, v, c_1, \dots, c_{k-1}\}}{2} \cup \{ua_1, va_k\} \cup \bigcup_{i=1}^{k-1} \{a_i c_i, c_i a_{i+1}\} \right) \setminus \{uv\}.$$

Note that the graphs H_3^k for $k \in \{3, \dots, 6\}$ equal the forbidden subgraphs H_3, H_4, H_5, H_6, H_7 introduced by Chang et al. and moreover, the graphs H_3 and H_6 are isomorphic. The

graphs H_3^k , for k at least 7, are minimal forbidden induced subgraphs for substar graphs and do not contain any graph as an induced subgraph from the list of Chang et al.

As already mentioned above, a chordal graph G has a \mathcal{T} -representation, that is, there is a tree T and a function $f : V(G) \rightarrow \mathcal{T}$, where \mathcal{T} is the set of all the subtrees of T and two distinct vertices v, w are adjacent if and only if $f(v) \cap f(w) \neq \emptyset$. We say the tuple (T, f) is a *tree representation* of G . It is well known and not difficult to see that one can choose T such that $f(V(G)) = T$ without increasing the diameter of any subtree in \mathcal{T} . Therefore, from now on we may assume that a tree representation (T, f) of a graph G has the property $f(V(G)) = T$.

Let (T, f) be a tree representation of a chordal graph G and $t \in V(T)$. Obviously, $f^{-1}(t)$ is a clique Q in G . Suppose that Q is not a maximal clique of G and let Q' be a maximal clique of G that contains Q . By the Helly property of subtrees of a tree (see for example [1]) and the maximality of Q' , there is a vertex $t' \in V(T)$ such that $f^{-1}(t') = V(Q')$. Let $P : tt_1 \dots t_k t'$ be the t, t' -path in T . Note that $V(Q) \subseteq f^{-1}(t_i)$ for every $i \in [k]$. Contracting the edge tt_1 to a vertex v_{tt_1} leads to a tree T' . Define f' by $f'^{-1}(s) = f^{-1}(s)$ if $s \in V(T) \setminus \{t, t_1\}$ and let $f'^{-1}(v_{tt_1}) = f^{-1}(t_1)$. Trivially (T', f') is a tree representation of G . Repeating this process leads to a tree representation (\tilde{T}, \tilde{f}) of G such that for every vertex $\tilde{t} \in V(\tilde{T})$ the vertex set $\tilde{f}^{-1}(\tilde{t})$ is a maximal clique in G . If there are two vertices t and t' of \tilde{T} such that $\tilde{f}^{-1}(t) = \tilde{f}^{-1}(t')$, then contracting all edges on the t, t' -path in \tilde{T} and repeating this process as often as possible leads to a tree representation (\hat{T}, \hat{f}) of G such that there is a bijection between the maximal cliques of G and the vertices of \hat{T} . Note that this process does not increase the diameter of a subtree in the tree representation induced by a vertex of G . Therefore, from now on we may assume that a tree representation (T, f) of a graph G has the property $f(V(G)) = T$ and there is a bijection between the maximal cliques of G and the vertices of T .

After these preliminaries, we proceed to our results.

Lemma 1. *The graphs in \mathcal{S} are not substar graphs.*

Proof: For contradiction we consider a substar representation (T, f) for some $G \in \mathcal{S}$ such that $f(v)$ does not contain a P_4 as a subgraph for every $v \in V(G)$.

First, let $G = H_1$. There are four maximal cliques $Q_1 = \{z, a, x\}$, $Q_2 = \{z, x, w\}$, $Q_3 = \{z, w, y\}$, and $Q_4 = \{z, y, b\}$ with corresponding vertices q_1, q_2, q_3 , and q_4 in T , respectively. Since $f(v)$ is connected for every $v \in V(G)$, $q_1 q_2 \in E(T)$, $q_2 q_3 \in E(T)$, and $q_3 q_4 \in E(T)$. Thus, T and $f(z)$ is a P_4 , which is a contradiction.

Now suppose that G is a graph of type H_2 . There are maximal cliques which contain the sets $\{z, a, x\}$, $\{z, x, u\}$, $\{z, v, y\}$, and $\{z, y, b\}$, respectively. Note that these maximal cliques are distinct, because for every pair Q and Q' of these sets there is a vertex of G in Q which is nonadjacent to a vertex in Q' . Suppose $f(z)$ is a star. If the subtree $f(x)$ contains the center vertex of $f(z)$, then $f(y)$ is disconnected, otherwise $f(x)$ is disconnected. Thus $f(z)$ is not a star, which is a contradiction.

Now we suppose $G = H_3^k$ for some $k \geq 3$. We denote by Q_u and Q_v the two maximal cliques $\{u, c_1, \dots, c_{k-1}\}$ and $\{v, c_1, \dots, c_{k-1}\}$ with corresponding vertices q_u and q_v in T , respectively. In addition, there are k maximal cliques $Q_1 = \{u, a_1, c_1\}$, $Q_i = \{c_{i-1}, a_i, c_i\}$ for $i \in \{2, \dots, k-1\}$, and $Q_k = \{c_{k-1}, a_k, v\}$ with corresponding vertices q_1, \dots, q_k in T . Since $f(v)$ is connected for every $v \in V(G)$, every vertex $w \in Q_u \cap Q_v$ is contained in every maximal clique that is associated with a vertex on the q_u, q_v -path in T . If $k = 3$ and $q_u q_v$ is not an edge in T , then $q_u q_2$ and $q_2 q_v$ are edges in T . Thus $q_u q_1$ is an edge and hence $f(c_1)$ is not a star, which is a contradiction. Since no maximal clique beside Q_u and Q_v contains $\{c_1, c_{k-1}\}$ if $k \geq 4$, we conclude that $q_u q_v$ is an edge in T . Since $f(u)$ and $f(v)$ is connected, we obtain $q_u q_1, q_v q_k \in E(T)$. Since $f(c_1)$ contains no P_4 , we conclude $q_u q_2 \in E(T)$. By repeating the last argument several times, we obtain $q_u q_i \in E(T)$ for every $i \in [k]$, and hence there is a cycle $q_u q_v q_k q_u$ in T . This is a contradiction to our assumption that T is a tree, which completes the proof. \square

We immediately proceed to our main result.

Theorem 2. *A chordal graph is a substar graph if and only if it has no induced subgraph in \mathcal{S} .*

Proof: By Lemma 1, a graph in the set \mathcal{S} is not a substar graph. We show that every graph that is not a substar graph contains a graph from the set \mathcal{S} . For contradiction, we assume that G is a chordal graph that is a minimal forbidden induced subgraph for substar graphs and does not belong to \mathcal{S} . Thus G is connected. Let (T, f) be a tree representation of G . We denote vertices of T by q with an optional additional subscript/superscript. The corresponding maximal clique in G is denoted by Q with the same optional additional subscript/superscript.

We first show that we may assume that every subtree has diameter at most 3, i.e. no subtree contains a P_5 , otherwise this is a contradiction to our choice of G . To see this, take a vertex u that is contained in only one maximal clique Q (every leaf of T contains such a vertex). Since all proper induced subgraphs of G are substar graphs, $G - u$ is a substar graph. Therefore, let (T', f') be a tree representation of $G - u$. Since $Q - u$ is a clique, its vertices are contained in some maximal clique Q' . Add to T' a vertex q'' adjacent to q' . Associate to q'' all vertices of Q . All subtrees in T' to which a vertex of $G - u$ is associated are still subtrees and may contain a P_4 but not a P_5 .

After this preliminary consideration, we proceed in the proof of Theorem 2. We choose a tree representation (T, f) of G such that the subtree $f(v)$ of some tree T does not contain a P_5 for every $v \in V(G)$ and the number of vertices v such that $f(v)$ has diameter 3 is minimal. Note that for every edge $qq' \in E(T)$, by our assumptions on T and the connectedness of G , we conclude $Q \setminus Q' \neq \emptyset$, and $Q \cap Q' \neq \emptyset$. Since G is not a substar graph, there is a vertex z of G such that $f(z)$ contains a $P_4 : q_1 q_2 q_3 q_4$. Let $q_2 q_3$ be the unique middle edge of all P_4 s in $f(z)$. Note that for q_1 and q_4 there are possibly many choices. We call a neighbor q of q_2 other than q_3 in T *good* if there is a vertex u of G such

that $u \in Q \cap Q_2$ but $u \notin Q_3$, and otherwise *bad*. Let k be a positive integer. We say that there is a *construction of order k* (with respect to q_2 and q_3) in G if the following holds:

- there is sequence of distinct vertices w_0, w_1, \dots, w_k of G such that $w_0 = z$,
- there are distinct neighbors q^1, \dots, q^k of q_2 other than q_3 in T

such that

- $f(z) \cap \{q^1, \dots, q^k\} = \{q^1\}$,
- $f(w_i) \cap \{q^1, \dots, q^k\} = \{q^i, q^{i+1}\}$ for $i \in [k-1]$,
- $f(w_k) \cap \{q^1, \dots, q^k\} = \{q^k\}$, and
- $q_3 \in f(w_i)$ if and only if $i \in \{0, \dots, k-1\}$.

We say that there is a *mirrored construction of order k* (with respect to q_2 and q_3) in G , if there is a construction of order k with respect to q_2 and q_3 in G after renaming q_i by q_{5-i} for $i \in [4]$. We say that we *switch* a neighbor q of q_2 other than q_3 , if we delete the edge qq_2 and add the edge qq_3 in T .

In the following part of the proof we show that there is a construction of order k and a mirrored construction of order ℓ with respect to q_2 and q_3 in G for some positive integers k and ℓ . Afterwards we show that this implies the existence of an induced subgraph in G from the set \mathcal{S} .

If there is a good neighbor of q_2 in $f(z)$, then there is a construction of order 1 in G . Therefore, we may assume that all neighbors of q_2 in $f(z)$ other than q_3 are bad and denote this set by V_1 . Now we simultaneously switch all vertices of V_1 and get a new tree representation (T_1, f_1) of G . If no diameter of a subtree $f(v)$ ($v \in V(G)$) increased by this switch, then the subtree $f_1(z)$ is a star and the number of vertices v of G such that $f_1(v)$ has diameter 3 decreased, which is a contradiction to our choice of T . Let $V'_1 \subseteq V_1$ be the set of vertices of V_1 that contains vertices w of G such that the diameter of $f_1(w)$ is larger than the diameter of $f(w)$. This implies that V'_1 is exactly the set of vertices of V_1 in T_1 , such that there is for every $q \in V'_1$ a vertex x_q of G with the following properties

- $q \in f(x_q)$,
- $q_2, q_3 \in f(x_q)$,
- there is a neighbor \tilde{q} of q_2 other than q_3 in T_1 such that $\tilde{q} \in f(x_q)$, and
- there is no neighbor q' of q_3 other than q_2 and other than the vertices of V_1 in T_1 such that $x_q \in Q'$.

Note that for every $q \in V'_1$ the diameter of $f(x_q)$ increases from 2 to 3 or from 3 to 4 by switching q . Let V_2 be set of all neighbors \hat{q} of q_2 in T_1 other than q_3 with the property that there is a vertex x of G such that

- $\hat{q} \in f(x)$,
- $\tilde{q} \in f(x)$ for some $\tilde{q} \in V'_1$, (note that these properties imply that $q_2, q_3 \in f(x)$), and
- there is no neighbor q' of q_3 other than q_2 and other than the vertices of V_1 in T_1 such that $q' \in f(x_q)$.

That means in V_2 are all vertices of T_1 that make problems because we switched the vertices of V_1 . If there is a good vertex in V_2 , then there is construction of order 2 in G . Therefore, we may assume all neighbors of q_2 in V_2 are bad. Now we simultaneously switch all vertices of V_2 and get a new tree representation (T_2, f_2) of G . Note that for every vertex v of G for which the diameter of $f_1(v)$ is larger than the diameter of $f(v)$ the diameter of $f_2(v)$ is as large as $f(v)$ again. If no diameter of a subtree $f_1(v)$ ($v \in V(G)$) increased by this switch, then the subtree $f_2(z)$ is a star and the number of vertices v of G such that $f_2(v)$ has diameter 3 decreased, which is a contradiction to our choice of T . Let $V'_2 \subseteq V_2$ be the set of vertices of V_2 that contain vertices w of G such that the diameter of $f_2(w)$ is larger than the diameter of $f_1(w)$. This implies that V'_2 is exactly the set of vertices of V_2 in T_2 , such that there is for every $q \in V'_2$ a vertex x_q of G with the following properties

- $q \in f(x_q)$,
- $q_2, q_3 \in f(x_q)$,
- there is a neighbor \tilde{q} of q_2 other than q_3 in T_2 such that $\tilde{q} \in f(x_q)$, and
- there is no neighbor q' of q_3 other than q_2 and other than the vertices of V_2 in T_2 such that $q' \in f(x_q)$.

Note that for every $q \in V'_2$ the diameter of $f(x_q)$ increases from 2 to 3 or from 3 to 4 by switching q .

It is easy to repeat this arguments for V_i, V'_i for $i > 2$, if V_i, V'_i is defined analogously. Extending the argumentation from above, we know, if there is a good vertex $v \in V_k$ for some $j \in \mathbb{N}$, then there is a construction of order j in G . However, if all vertices in V_j are bad, then V_{j+1} is nonempty. Since the number of vertices is finite, there is a good vertex in V_k for some positive integer k . Thus we obtain a construction of order k in G . By symmetry, we can argue analogously, if we swap the names of q_2 and q_3 . Thus we obtain a construction of order k and a mirrored construction of order ℓ for some positive integers k, ℓ with respect to q_2 and q_3 in G .

Suppose $k = 1$ and $\ell = 1$, then let

- x be the vertex that is associated with w_1 in the construction of order 1,
- y be the vertex that is associated with w_1 in the mirrored construction of order 1,
- $a \in Q_1 \setminus Q_2$,

- $b \in Q_4 \setminus Q_3$,
- $u \in Q_2 \setminus Q_1$, and
- $v \in Q_3 \setminus Q_4$.

If $u = v$, then these vertices, together with z , induce the graph H_1 , otherwise the graph H_1 or a graph of type H_2 .

Suppose $k + \ell > 2$, then $H_3^{k+\ell}$ is an induced subgraph of G . To see this, let

- $z = c_\ell$,
- $c_{\ell+i}$ be the vertex that is associated with w_i in the construction of order k for $i \in [k-1]$,
- v be the vertex that is associated with w_k in the construction of order k ,
- c_j be the vertex that is associated with $w_{\ell-j}$ in the mirrored construction of order ℓ for $j \in [\ell-1]$,
- u be the vertex that is associated with w_ℓ in the mirrored construction of order ℓ ,
- $a_{\ell+i}$ be a vertex in $Q^i \setminus Q_2$ for $i \in [k]$ where Q^i is associated with the construction of order k , and
- $a_{\ell+1-j}$ be a vertex in $Q^j \setminus Q_3$ for $j \in [\ell]$ where Q^j is associated with the mirrored construction of order ℓ .

This completes the proof of Theorem 2. \square

References

- [1] A. Brandstädt, V.B. Le and J.P. Spinrad, Graph Classes: A Survey, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.
- [2] M.R. Cerioli and J.L. Szwarcfiter, Characterizing Intersection Graphs of Substars of a Star, *Ars Combin*, **79**, 21-31 (2006).
- [3] Y.-W. Chang, M.S. Jacobson, C.L. Monma and D.B. West, Subtree and substar intersection numbers, *Discrete Appl. Math.* **44**, 205-220 (1993).
- [4] P.C. Fishburn. Interval Orders and Interval Graphs. John Wiley & Sons (1985).
- [5] F. Gavril, A recognition algorithm for the intersection graphs of paths in trees, *Discrete Math.*, **23**, 211-227 (1978).
- [6] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, *J. Combinatorial Theory Ser. B* **16**, 47-56 (1974).
- [7] C.L. Monma and V.K. Wei, Intersection graphs of paths in a tree, *J. Combin. Theory Ser. B* **41**, 141-181 (1986).